Journal of Guidance, Control, and Dynamics
Vol. 34, No. 2, March–April 2011

Lyapunov Differential Equation Approach to Elliptical Orbital Rendezvous with Constrained Controls

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DOI: 10.2514/1.52372

This paper is concerned with spacecraft rendezvous with target spacecraft in an arbitrary elliptical orbit. With three independent control accelerations being the control of the resulting linearized Tschauner–Hempel equations, the spacecraft rendezvous problem can be reformulated as a regulation problem with controls of bounded magnitude and energy. A parametric Lyapunov differential equation approach is proposed in this paper to solve this constrained regulation problem. After establishing the fact that the Tschauner–Hempel equations are both null controllable with controls of bounded magnitude and energy, this paper proves that the proposed linear periodic controller semiglobally stabilizes the system. Equivalently, for any fixed initial conditions, the magnitude and energy of the control can be made as small as desired by tuning some free parameters in the feedback laws. In comparison with the existing quadratic-regulation-based approach, which requires solutions to nonlinear Riccati differential equations, the new approach requires only the solution of linear periodic Lyapunov differential equations, which are investigated in the paper by using the periodic generator approach. Numerical simulations of the nonlinear model of the spacecraft rendezvous instead of a linearized one show that both the magnitude and energy of the control can be reduced to an arbitrarily small level by reducing the values of some parameters in the controller and that the rendezvous mission can be accomplished satisfactorily.

Nomenclature

a	=	semimajor axis of target orbit
A_i, $\mathcal{A}_i$	=	state matrices for in-plane motion
A_o, $\mathcal{A}_o$	=	state matrices for out-of-plane motion
$\boldsymbol{a}_f$	=	acceleration due to thrust forces on chaser
B_i, $\mathcal{B}_i$	=	control matrices for in-plane motion
B_o, $\mathcal{B}_o$	=	control matrices for out-of-plane motion
$\mathcal{C}(A)$	=	characteristic multiplier set for periodic matrix A
E	=	eccentric anomaly
e	=	eccentricity of target orbit
h	=	orbital angular momentum of target orbit
k	=	$\mu/h^{3/2}$
K_i, K_o	=	feedback gains for in-plane and out-of-plane motions
L_i, L_o	=	transformation matrices
P_i, P_o	=	solutions to parametric Riccati differential equations
$\boldsymbol{R}$	=	vector from center of gravity to target spacecraft
$\boldsymbol{r}$	=	vector from target spacecraft to chaser spacecraft
T	=	period of target orbit
t	=	time
$\mathrm{tr}(A)$	=	trace of square matrix A
T_s	=	settling time
$\boldsymbol{u}_i$, $\boldsymbol{v}_i$	=	control vectors of in-plane motion
$\boldsymbol{u}_o$, $\boldsymbol{v}_o$	=	control vectors of out-of-plane motion
$\Vert\boldsymbol{u}\Vert_{L_2}$	=	L_2 norm of vector $\boldsymbol{u}(t)$ (i.e., the square root of $\int_{t_0}^{\infty}\Vert\boldsymbol{u}(t)\Vert^2\,\mathrm{d}t$)
$\Vert\boldsymbol{u}\Vert_{L_\infty}$	=	L_∞ norm of vector $\boldsymbol{u}(t)$ (i.e., $\sup_t\Vert\boldsymbol{u}(t)\Vert$)
W_i, W_o	=	solutions to Lyapunov differential equations
(x, y, z)	=	rotating coordinate frame fixed at target spacecraft
$\boldsymbol{\zeta}$, $\boldsymbol{\xi}$	=	state vectors of Tschauner–Hempel equations
$\boldsymbol{\zeta}_i$, $\boldsymbol{\xi}_i$	=	state vectors of in-plane motion
$\boldsymbol{\zeta}_o$, $\boldsymbol{\xi}_o$	=	state vectors of out-of-plane motion
θ	=	true anomaly
$\lambda(A)$	=	eigenvalue set of matrix A
μ	=	gravity constant
ρ	=	$1 + e\cos(\theta)$
$\Phi_i(\theta, \vartheta)$	=	state transition matrix of in-plane motion
$\Phi_o(\theta, \vartheta)$	=	state transition matrix of out-of-plane motion
ω	=	orbital rate of target spacecraft

I. Introduction

Many astronautic missions, such as repairing, saving, intercepting, docking, large-scale structure assembling, and satellite networking, rely heavily on successful rendezvous [1]. During the past few decades, considerable attention has been paid to the spacecraft rendezvous control problem in the literature. Considering a target spacecraft in a circular or elliptical orbit and another chaser spacecraft in its neighborhood, the relative motion of the chaser with respect to the target can be described by autonomous nonlinear differential equations for which the linearized equations are known as Hill–Clohessy–Wiltshire (H–C–W) equations [2] or Tschauner–Hempel (T–H) equations [3]. The H–C–W equations are time invariant while the T–H equations are periodic. Yet, both of them possess periodic solutions that constitute relative orbits of the chaser and are useful for passive rendezvous and formation flight [1]. Hence, these efficient equations are the basic tools and foundation in rendezvous (see, for example, [4–8]).

Received 13 September 2010; revision received 4 December 2010; accepted for publication 7 December 2010.

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Spacecraft rendezvous has attracted renewed interest in recent years as a result of new developments in control theory and technologies. Many advanced methods have been applied to solve the rendezvous control problem with different requirements. For example, the Huber filter approach to addressing the problem of radar- or laser-based robust rendezvous navigation is investigated in [9]; adaptive control theory was applied to solve the rendezvous and docking problem in [10]; in [11], the neural-network-based controller is proposed for rendezvous maneuvers; autonomous rendezvous for a cooperative target or noncooperative target based on relative orbit elements is considered in [12]; multiobjective robust H_∞ control of spacecraft rendezvous in a circular orbit is studied in [13]; and the optimal multiobjective linearized impulsive rendezvous problem is investigated in [14]. For more related work, see the references cited in the aforementioned papers.

Constraints such as actuator saturation and power limitation exist in every practical control system. Constraints are not only the source of performance degradation but also the source of instability. Hence, in the past several decades, many researchers have paid significant attention to constrained control problems (see [15–23] and the references therein). Practical constraints have also attracted much attention in the area of spacecraft rendezvous control. One of the most frequently considered constraints is power limitation. Power-limited spacecraft have propulsion systems that have an upper bound on the exhaust power that the engine can supply [20,24–26]. Another constraint that is also frequently studied is saturation nonlinearity in the actuator: say, thrust saturation (see, for example, [27,28]). It is also possible to consider these two constraints simultaneously [29–31]. Most of the solutions to this problem are based on optimal control theory. The resulting controllers are, however, complicated and hard to compute and implement.

In this paper, the rendezvous problem with the target spacecraft in a general elliptical orbit is considered. With three independent continuous control accelerations (or thrusts) being used as the control signals to the T–H equations, the spacecraft rendezvous problem can be formulated as a regulation problem with controls of bounded magnitude and energy. A parametric Lyapunov differential equation (PLDE) approach is proposed for designing a linear periodic controller for the problem. By showing that the T–H equations are asymptotically null controllable with bounded controls (to be defined later) and recognizing that they are also null controllable with vanishing energy (NCVE) (to be introduced later; see also [20]), it is established that the resulting controller achieves semiglobal stabilization of the relative motion with magnitude and/or energy-bounded controls by tuning some free parameters in the feedback laws. Semiglobal stabilization means that the initial condition can be as large as necessary as long as it is bounded. Semiglobal stabilization is more reasonable than global stabilization in the spacecraft rendezvous problem, as the linearized relative motion (say, T–H equations) are applicable only when the relative error (the states of the system) is not too large (see, for example, [3]). Roughly speaking, for any fixed initial condition, the magnitude and control energy in rendering the initial condition to the origin can be made as small as needed by reducing the values of some parameters in the PLDEs. One of the main advantages of the proposed PLDE-based approach over the existing approaches (especially those based on quadratic regulation theory and requiring solutions to nonlinear periodic differential Riccati equations) is that its resulting controller is easy to implement, since only a linear periodic differential equation is required to be solved. An effective periodic generator approach is also established to solve such a class of linear periodic equations. The proposed approach is applied to a rendezvous system in which the target spacecraft is in an eccentric orbit of geostationary transfer orbit type. It should be emphasized that the designed linear periodic controller is applied directly to the original nonlinear equations of the relative motion. The performances, magnitude of the control, and control energy of the resulting closed-loop system are calculated with different values of the free parameters in the controller. It is observed that a tradeoff between the system performances and the magnitude (energy) of the controls exists in choosing the free parameters.

The remainder of this paper is organized as follows. Section II presents the relative motion of the spacecraft rendezvous problem (Sec. II.A) and investigates its properties, especially the controllability by magnitude and/or energy-bounded controls (Sec. II.B), by carrying out the Jordan canonical form of the T–H equations. The PLDE-based approach is then introduced in Sec. III to solve the spacecraft rendezvous problem with constraints. The theoretical results and an algorithm are, respectively, given in Secs. III.A and III.B. A numerical example is worked out in Sec. IV to show the effectiveness of the proposed methodology.

II. Relative Motion and Properties

A. Tschauner–Hempel Equations and Simplification

Assume that the target spacecraft is in an eccentric orbit. Denote as $\boldsymbol{r}$ and $\boldsymbol{R}$, respectively, the vector from the target spacecraft to the chaser spacecraft and the vector from the center of gravity to the target spacecraft. The relative motion of the chaser spacecraft in the inertial frame can be captured by [32]

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\mu\left(\frac{\boldsymbol{R}+\boldsymbol{r}}{|\boldsymbol{R}+\boldsymbol{r}|^3} - \frac{\boldsymbol{R}}{|\boldsymbol{R}|^3}\right) + \boldsymbol{a}_f \tag{1}$$

where μ is the gravity constant and $\boldsymbol{a}_f$ is the acceleration vector due to thrust forces on the chaser spacecraft. Consider the target-orbital rotating coordinate system (or target-orbital coordinate system, for short) x-y-z, as shown in Fig. 1, where the origin is fixed at the center of mass of the target, and the y axis is normal to the orbital plane. For an arbitrary variable v, the symbol $\dot{v}$ denotes the derivative with respect to time in such a rotating coordinate system. Then, with the notation $\boldsymbol{r} = [x \quad y \quad z]^T$, Eq. (1) can be written as [32]

$$\begin{bmatrix} \ddot{x} \\ \ddot{z} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 2\omega\dot{z} + \dot{\omega}z + \omega^2 x - \frac{\mu x}{|\boldsymbol{R}+\boldsymbol{r}|^3} \\ \omega^2 z - 2\omega\dot{x} - \dot{\omega}x - \mu\left(\frac{z-R}{|\boldsymbol{R}+\boldsymbol{r}|^3} + \frac{1}{R^2}\right) \\ -\frac{\mu y}{|\boldsymbol{R}+\boldsymbol{r}|^3} \end{bmatrix} + \boldsymbol{a}_f \tag{2}$$

where $|\boldsymbol{R}| = R$, $|\boldsymbol{r}| = r$, and $|\boldsymbol{R}+\boldsymbol{r}|^2 = x^2 + y^2 + (z-R)^2$, and ω is the orbital rate of the rotating coordinate system.

Let h be the orbital angular momentum of the target. Then, $R^2\omega = h =$ constant. Let $e \in [0, 1)$ be the eccentricity of the orbit, θ be the true anomaly, $\rho = 1 + e\cos(\theta)$, and

$$k = \frac{\mu}{h^{\frac{3}{2}}} = \text{constant} \tag{3}$$

Then, it is easy to see that

$$\omega = \frac{h}{R^2} = k^2\rho^2 \tag{4}$$

The true anomaly θ and the eccentric anomaly E satisfy the following well-known equation:

$$\sin(E) = \frac{\sqrt{1-e^2}\sin(\theta)}{1+e\cos(\theta)}, \qquad \cos(E) = \frac{e+\cos(\theta)}{1+e\cos(\theta)} \tag{5}$$

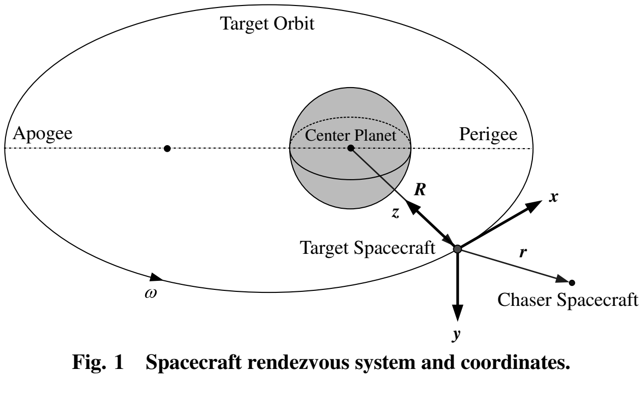

Fig. 1 Spacecraft rendezvous system and coordinates.

If the target spacecraft is at perigee at $t = 0$ (namely, both the true anomaly θ and eccentric anomaly E are zero at $t = 0$), then the eccentric anomaly E and time t satisfy Kepler's equation:

$$t = \frac{T}{2\pi}[E - e\sin(E)] \tag{6}$$

If the distance between the chaser and the target is much smaller than the distance between the target and the center of the gravity field (i.e., $R \gg r$), then the nonlinear system in Eq. (2) can be linearized at the origin as [32]

$$\begin{bmatrix} \ddot{x} \\ \ddot{z} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -k\omega^{3/2}x + 2\omega\dot{z} + \dot{\omega}z + \omega^2 x \\ 2k\omega^{3/2}z - 2\omega\dot{x} - \dot{\omega}x + \omega^2 z \\ -k\omega^{3/2}y \end{bmatrix} + \boldsymbol{a}_f \tag{7}$$

which are known as the T–H equations. It follows from Eq. (7) that the in-plane motion (i.e., the x-z subsystem) and the out-of-plane motion (i.e., the y subsystem) are independent. Therefore, they can be considered separately.

With the notation $\boldsymbol{a}_f = [\,a_f^x \quad a_f^z \quad a_f^y\,]^T$, and choosing

$$\boldsymbol{\xi}_i(t) = [\,x(t) \quad z(t) \quad \dot{x}(t) \quad \dot{z}(t)\,]^T, \qquad \boldsymbol{v}_i(t) = [\,a_f^x(t) \quad a_f^z(t)\,]^T \tag{8}$$

the in-plane motion

$$\begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -k\omega^{3/2}x + 2\omega\dot{z} + \dot{\omega}z + \omega^2 x \\ 2k\omega^{3/2}z - 2\omega\dot{x} - \dot{\omega}x + \omega^2 z \end{bmatrix} + \begin{bmatrix} a_f^x \\ a_f^z \end{bmatrix} \tag{9}$$

can be written as

$$\dot{\boldsymbol{\xi}}_i(t) = \mathcal{A}_i(t)\boldsymbol{\xi}_i(t) + \mathcal{B}_i(t)\boldsymbol{v}_i(t) \tag{10}$$

where $\mathcal{A}_i(t)$ and $\mathcal{B}_i(t)$ are given, respectively, by

$$\mathcal{A}_i(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 - k\omega^{\frac{3}{2}} & \dot{\omega} & 0 & 2\omega \\ -\dot{\omega} & \omega^2 + 2k\omega^{3/2} & -2\omega & 0 \end{bmatrix}$$

$$\mathcal{B}_i(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{11}$$

It follows that the system described by Eq. (10) is periodic with period T, the period of the target orbit. Similarly, with

$$\boldsymbol{\xi}_o(t) = [\,y(t) \quad \dot{y}(t)\,]^T, \qquad \boldsymbol{v}_o(t) = a_f^y(t)$$

the out-of-plane motion can be expressed as

$$\dot{\boldsymbol{\xi}}_o(t) = \mathcal{A}_o(t)\boldsymbol{\xi}_o(t) + \mathcal{B}_o(t)\boldsymbol{v}_o(t) \tag{12}$$

where

$$\mathcal{A}_o(t) = \begin{bmatrix} 0 & 1 \\ -k\omega^{3/2} & 0 \end{bmatrix}, \qquad \mathcal{B}_o(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The system described by Eq. (12) is also a linear T-periodic system.

The method in [3] will be used to simplify the systems described by Eqs. (10) and (12). In this method, the true anomaly θ is used as an independent variable instead of the time variable t. Denote the derivative of a variable v with respect to θ as v', and consider the state transformation

$$[\,\hat{x}(\theta) \quad \hat{y}(\theta) \quad \hat{z}(\theta)\,] = \rho(\theta)[\,x(t) \quad y(t) \quad z(t)\,] \tag{13}$$

Choose a new state vector $\boldsymbol{\zeta}(\theta) = [\,\boldsymbol{\zeta}_i^T(\theta) \quad \boldsymbol{\zeta}_o^T(\theta)\,]^T$ and a control vector $\boldsymbol{u}(\theta) = [\,\boldsymbol{u}_i^T(\theta) \quad \boldsymbol{u}_o^T(\theta)\,]^T$ with

$$\boldsymbol{\zeta}_i(\theta) = [\,\hat{x}(\theta) \quad \hat{z}(\theta) \quad \hat{x}'(\theta) \quad \hat{z}'(\theta)\,]^T$$
$$\boldsymbol{\zeta}_o(\theta) = [\,\hat{y}(\theta) \quad \hat{y}'(\theta)\,]^T, \qquad \boldsymbol{u}_i(\theta) = [\,a_f^x(\theta) \quad a_f^z(\theta)\,]^T$$
$$\boldsymbol{u}_o(\theta) = a_f^y(\theta)$$

It can be verified that $\boldsymbol{\zeta}_i(\theta) = L_i(\theta)\boldsymbol{\xi}_i(t)$ with

$$L_i(\theta) = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ \rho' & 0 & \frac{\rho}{\omega} & 0 \\ 0 & \rho' & 0 & \frac{\rho}{\omega} \end{bmatrix} \tag{14}$$

The system described by Eq. (10) can then be transformed into

$$\boldsymbol{\zeta}_i'(\theta) = A_i(\theta)\boldsymbol{\zeta}_i(\theta) + B_i(\theta)\boldsymbol{u}_i(\theta) \tag{15}$$

where $A_i(\theta)$ and $B_i(\theta)$ are given, respectively, by

$$A_i(\theta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & \frac{3}{\rho} & -2 & 0 \end{bmatrix}, \qquad B_i(\theta) = \frac{1}{k^4\rho^3}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{16}$$

Notice that both $A_i(\theta)$ and $B_i(\theta)$ are periodic matrices with period 2π.

Similarly, under the state transformation described by Eq. (13), that is,

$$\boldsymbol{\zeta}_o(\theta) = \begin{bmatrix} \rho & 0 \\ \rho' & \frac{\rho}{\omega} \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \triangleq L_o(\theta)\boldsymbol{\xi}_o(t) \tag{17}$$

the out-of-plane motion described by Eq. (12) can be simplified as

$$\boldsymbol{\zeta}_o'(\theta) = A_o(\theta)\boldsymbol{\zeta}_o(\theta) + B_o(\theta)\boldsymbol{u}_o(\theta) \tag{18}$$

where $A_o(\theta)$ and $B_o(\theta)$ are given, respectively, as

$$A_o(\theta) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B_o(\theta) = \frac{1}{\rho^3 k^4}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{19}$$

Notice that the system matrix is time invariant.

The whole rendezvous process can be described by the transformation of state vectors $\boldsymbol{\zeta}(\theta)$ from nonzero initial states $\boldsymbol{\zeta}(\theta_0)$ to the terminal state $\boldsymbol{\zeta}(\theta_f) = 0$, where $\theta_f = \theta(t_f)$, with t_f being the rendezvous time. To meet the requirements of actual conditions, especially the limited energy (power) of the actuator and the maximal control signals that the actuator can generate, in this paper, controllers will be designed to solve the preceding problem by imposing the bounds

$$\|\boldsymbol{u}\|_{L_\infty} \le \mu_\infty$$

and

$$\|\boldsymbol{u}\|_{L_2} \le \mu_2$$

where μ_∞ and μ_2 are some given positive scalars.

B. Properties of Tschauner–Hempel Equations

Some basic results for periodic systems are introduced first. Consider the following general periodic system:

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t) \tag{20}$$

where $A(t) \in \mathbf{R}^{n\times n}$ and $B(t) \in \mathbf{R}^{n\times m}$ are ϖ periodic with a given $\varpi > 0$. Denote the state transition matrix of the system by $\Phi(t, \tau)$.

Definition 1 [33]: A complex number η is said to be the characteristic multiplier of the ϖ-periodic matrix $A(t)$ if there exists a ϖ-periodic differentiable vector function $\boldsymbol{v}(t)$: $\mathbf{R} \to \mathbf{C}^n$ such that $\boldsymbol{v}(t) \ne 0$, such that $t \in \mathbf{R}$ and

$$\boldsymbol{\xi}(t) = \boldsymbol{v}(t)\eta^{t/\varpi}, \quad \forall\ t \in \mathbf{R}$$

is a solution to a system described by Eq. (20) with initial time $t = 0$ and $\boldsymbol{u}(t) = 0$, such that $t \in \mathbf{R}$. Moreover, $\boldsymbol{v}(t)$ is referred to as a right characteristic vector of matrix $A(t)$.

Hereafter, the set of characteristic multipliers for $A(t)$ is denoted by $\mathcal{C}[A(t)]$.

Definition 2 [34]: Let $T(t)$: $\mathbf{R} \to \mathbf{R}^{n\times n}$ be a ϖ-periodic matrix. Then, $\boldsymbol{x}(t) = T(t)\boldsymbol{z}(t)$, where $\boldsymbol{z}(t)$ is the state vector of the transformed system, is said to be a Lyapunov transformation if $T(t)$ is differentiable, and $T(t)$, $\dot{T}(t)$, and $T^{-1}(t)$ are bounded for all $t \in \mathbf{R}$. In this case, $T(t)$ is referred to as a ϖ-Lyapunov transformation matrix.

Proposition 1: The ϖ-periodic system described by Eq. (20) is ANCBC if, and only if, $[A(t), B(t)]$ is completely controllable and $|\mu| \le 1$, such that $\mu \in \mathcal{C}[A(t)]$.

It is interesting to notice that the conditions for ANCBC happen to be the same as those for NCVE [20].

Now, consider the properties of the T–H equations. Denote

$$J(\theta) = \int_0^\theta \frac{1}{\rho^2(\tau)}\,\mathrm{d}\tau \tag{21}$$

Then, according to the results in [32], it can be verified that

$$F_i(\theta) = \begin{bmatrix} 1 & -[1+\rho(\theta)]\cos(\theta) & [1+\rho(\theta)]\sin(\theta) & 3\rho^2(\theta)J(\theta) \\ 0 & \rho(\theta)\sin(\theta) & \rho(\theta)\cos(\theta) & 2-3e\rho(\theta)\sin\theta(\theta) \\ 0 & 2\rho(\theta)\sin(\theta) & 2\rho(\theta)\cos(\theta)-e & 3-6e\rho(\theta)\sin(\theta)J(\theta) \\ 0 & \cos(\theta)+e\cos(2\theta) & -[\sin(\theta)+e\sin(2\theta)] & F_{i44}(\theta) \end{bmatrix} \tag{22}$$

is a solution to a system described by Eq. (15) in the case of $\boldsymbol{u}_i(\theta) = 0$, where

$$F_{i44}(\theta) = -3e[(\cos(\theta) + e\cos(2\theta))J(\theta) + \sin(\theta)\rho^{-1}(\theta)]$$

Moreover, it is readily verified that $\det[F_i(\theta)] = e^2 - 1 \ne 0$, such that $\theta \in \mathbf{R}$. Hence, the state transition matrix of the system described by Eq. (15) can be expressed as

$$\Phi_i(\theta,\vartheta) = F_i(\theta)F_i^{-1}(\vartheta) \tag{23}$$

The following result is concerned with the real Jordan canonical form of the linearized in-plane motion described by Eq. (15). The proof is provided in the Appendix for clarity.

Theorem 1: The real Jordan canonical form of the 2π-periodic system described by Eq. (15) is

$$J_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

namely, the periodic system described by Eq. (15) is marginally unstable.

The preceding result gives a profound understanding of the linearized in-plane motion described by Eq. (15). It follows from this result and Lemma 1 that $\mathcal{C}[A_i(\theta)] = \{1, 1, 1, 1\}$, which was also shown in [20] for the original system described by Eq. (11). However, the Jordan canonical form can provide more information than the characteristic multiplier set, as has been mentioned earlier.

Corollary 1: The linearized in-plane motion described by Eq. (15) is both ANCBC and NCVE.

Proof: It follows from Theorem 1 that $\mathcal{C}[A_i(\theta)] = \{1, 1, 1, 1\}$. Hence, it remains to be verified that $[A_i(\theta), B_i(\theta)]$ is completely controllable. In fact, direct manipulation gives

$$\begin{aligned} &[\,B_i(\theta) \quad B_i'(\theta) - A_i(\theta)B_i(\theta)\,] \\ &= \frac{1}{k^4\rho^4}\begin{bmatrix} 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & -\rho \\ \rho & 0 & 3e\sin(\theta) & -2\rho \\ 0 & \rho & 2\rho & 3e\sin(\theta) \end{bmatrix} \end{aligned}$$

which is nonsingular for all $\theta \in \mathbf{R}$. Hence, according to Theorem 4.5.3 in [34], the system $[A_i(\theta), B_i(\theta)]$ is completely controllable. The proof is completed. □

At the end of this subsection, the out-of-plane motion will be considered. Notice that $A_o(\theta)$ is time invariant; hence,

Definition 3 [33,35]: A (real) Jordan matrix $J \in \mathbf{C}^{n\times n}$ ($\mathbf{R}^{n\times n}$) is said to be the (real) Jordan canonical form of the ϖ-periodic matrix $A(t)$ if there exists a (real) ϖ-Lyapunov transformation matrix $V(t)$: $\mathbf{R} \to \mathbf{C}^{n\times n}$ ($\mathbf{R}^{n\times n}$) such that

$$\dot{V}(t) = A(t)V(t) - V(t)J$$

Moreover, $V(t)$ is referred to as the characteristic vector matrix of $A(t)$.

Lemma 1 [35]: Let the Jordan canonical form of the ϖ-periodic matrix $A(t)$ be J. Then,

$$\mathcal{C}(A(t)) = \lambda(e^{J\varpi}) = \lambda[\Phi(t+\varpi, t)]$$

Moreover, the ϖ-periodic system described by Eq. (20) is asymptotically stable if, and only if, all of the characteristic multipliers of $A(t)$ are located strictly inside the unit circle or, equivalently, all the eigenvalues of its associated Jordan canonical form J are located on the open left half plane.

It should be pointed out that the Jordan canonical form can provide more information of periodic systems than its characteristic multiplier set.

Definition 4 [21]: The ϖ-periodic system described by Eq. (20) is said to be asymptotically null controllable by bounded controls (ANCBC) if, for any initial condition $\boldsymbol{x}_0 \in \mathbf{R}^n$, there exists a time $t_1 > t_0$ and control $\boldsymbol{u}(t)$, $t \in [t_0, t_1]$, such that

$$\sup_{t\in[t_0,t_1]}\|\boldsymbol{u}(t)\| \le 1$$

and the solution $\boldsymbol{x}(t)$ to the system described by Eq. (20) with initial condition $\boldsymbol{x}(t_0) = x_0$ satisfies $\boldsymbol{x}(t_1) = 0$.

Definition 5 [20]: The system described by Eq. (20) is said to be NCVE if, for each initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, there exists a sequence of pairs $(T_N, \boldsymbol{u}_N)$, $0 \le T_N < \infty$, and $\boldsymbol{u}_N \in L_2(0, T_N, \mathbf{R}^m)$ with

$$L_2(0, T, \mathbf{R}^m) \triangleq \left\{ \boldsymbol{f}(t)\colon [0, T] \to \mathbf{R}^m \,\middle|\, \int_0^T \|\boldsymbol{f}(t)\|^2\,\mathrm{d}t < \infty \right\}$$

such that $\boldsymbol{x}(T_N, x_0, \boldsymbol{u}_N) = 0$ and

$$\lim_{N\to\infty}\int_0^{T_N}\|\boldsymbol{u}_N(t)\|^2\mathrm{d}t = 0$$

Roughly speaking, ANCBC and NCVE mean, respectively, that any bounded fixed initial condition can be driven to the origin with the control having an arbitrarily small magnitude and arbitrarily small energy. Regarding the test for ANCBC property, the following result has been proved in [21].

$$\Phi_o(\theta, \theta_0) = e^{A_o(\theta-\theta_0)} = \begin{bmatrix} \cos(\theta-\theta_0) & \sin(\theta-\theta_0) \\ -\sin(\theta-\theta_0) & \cos(\theta-\theta_0) \end{bmatrix} \quad (25)$$

It then follows from Lemma 1 that

$$\mathcal{C}[A_o(\theta)] = \lambda[\Phi_o(\theta_0 + 2\pi, \theta_0)] = \{1, 1\}$$

On the other hand, it can be determined that

$$[\,B_o(\theta) \quad B_o'(\theta) - A_o(\theta)B_o(\theta)\,] = \frac{1}{\rho^4 k^4}\begin{bmatrix} 0 & -\rho \\ \rho & 3e\sin(\theta) \end{bmatrix}$$

which is nonsingular for all $\theta \in \mathbf{R}$. Hence, similar to Corollary 1, the out-of-plane motion is also both ANCBC and NCVE.

III. Parametric-Lyapunov-Differential-Equation-Based Approach to Orbital Rendezvous Control

A. Parametric-Lyapunov-Differential-Equation-Based Semiglobal Stabilization with Constrained Controls

In this section, the PLDE-based approach will be used to stabilize the in-plane system described by Eq. (15) and the out-of-plane system described by Eq. (18) with controls of bounded magnitude and bounded energy. Consider the following 2π-periodic Riccati differential equation associated with the in-plane motion described by Eq. (15):

$$\begin{aligned} -P_i'(\theta) &= A_i^T(\theta)P_i(\theta) + P_i(\theta)A_i(\theta) + \gamma_i P_i(\theta) \\ &\quad - P_i(\theta)B_i(\theta)R_i^{-1}(\theta)B_i^T(\theta)P_i(\theta) \end{aligned} \quad (26)$$

where $R_i(\theta)$ is an arbitrary 2π-periodic positive definite matrix and $\gamma_i \in (0, D_i]$, with $D_i > 0$ being an arbitrary scalar. Since $\mathcal{C}[A_i(\theta)] = \{1, 1, 1, 1\}$, the following result regarding solutions to Eq. (26) has been proved in [36] for more general periodic systems. For completeness, a short proof is provided in the Appendix.

Lemma 2: Consider the periodic Riccati differential equation described by Eq. (26).

1) It has a unique 2π-periodic positive definite solution $P_i(\theta) = P_i(\theta, \gamma_i) = W_i^{-1}(\theta)$, with $W_i(\theta) = W_i(\theta, \gamma_i)$ being the unique 2π-periodic positive definite solution to the following 2π-periodic PLDE:

$$W_i'(\theta) = W_i(\theta)A_i^T(\theta, \gamma_i) + A_i(\theta, \gamma_i)W_i(\theta) - B_i(\theta)R_i^{-1}(\theta)B_i^T(\theta) \quad (27)$$

in which $A_i(\theta, \gamma_i) = A_i(\theta) + \frac{1}{2}\gamma_i I_4$.

2) Let $P_i(\theta)$ be the 2π-periodic positive definite solution. Then, the periodic linear state feedback $\boldsymbol{u}_i(\theta) = K_i(\theta)\boldsymbol{\zeta}_i(\theta)$, where

$$K_i(\theta) = -R_i^{-1}(\theta)B_i^T(\theta)P_i(\theta) \quad (28)$$

stabilizes the system described by Eq. (15).

3) Let $P_i(\theta)$ be the 2π-periodic positive definite solution. Then,

$$\lim_{\gamma_i \to 0^+} P_i(\theta) = 0, \quad \forall\ \theta \in \mathbf{R} \quad (29)$$

Remark 1: Lemma 2 is also valid if the parameter γ_i in Eq. (26) is a function of θ and is parameterized by an independent variable ε. For example, as has been studied in [36], one can let $\gamma_i = \gamma_i(\theta, \varepsilon)$: $\mathbf{R} \times (0, D_i] \to \mathbf{R}$ be continuous on θ, uniformly continuous on ε, 2π periodic with respect to θ, and satisfy some other further conditions. Here, γ_i is chosen to be independent of θ for clarity.

It follows that, to stabilize the 2π-periodic system described by Eq. (15), only a linear differential equation in the form of Eq. (27) needs to be solved. Although the analytical solution can be obtained as [36]

$$W_i(\theta) = \int_\theta^\infty \Phi_i(\theta, s)B_i(s)R_i^{-1}(s)B_i^T(s)\Phi_i^T(\theta, s)e^{-\gamma_i(s-\theta)}\,\mathrm{d}s \quad (30)$$

the integration in the solution is hard to compute. A detailed discussion regarding the numerical computation of the solution to Eq. (27) will be given later.

Properties of the PLDE-based stabilizing controller described by Eq. (28) will consequently be discussed.

Theorem 2: For any given positive scalar μ_∞ and bounded set $\mathcal{X} \subset \mathbf{R}^4$, there exists a scalar $\gamma_i^* \in (0, D_i]$ such that the closed-loop system consisting of the periodic system described by Eq. (15) and the control

$$\boldsymbol{u}_i(\theta) = K_i(\theta)\boldsymbol{\zeta}_i(\theta) \quad (31)$$

with $\gamma_i \in (0, \gamma_i^*]$ are asymptotically stable with $\mathcal{X}$ contained in the domain of attraction and such that

$$\|\boldsymbol{u}_i\|_{L_\infty} \le \mu_\infty \quad (32)$$

Proof: The asymptotic stability of the closed-loop system follows from Lemma 2. Only Eq. (32) will be shown in the following. Rewrite the closed-loop system as

$$\boldsymbol{\zeta}_i'(\theta) = [A_i(\theta) + B_i(\theta)K_i(\theta)]\boldsymbol{\zeta}_i(\theta), \qquad \boldsymbol{\zeta}_i(\theta_0) \in \mathcal{X} \subset \mathbf{R}^4 \quad (33)$$

Notice that, by using Eq. (26), the derivative of $V[\boldsymbol{\zeta}_i(\theta)] = \boldsymbol{\zeta}_i^T(\theta)P_i(\theta)\boldsymbol{\zeta}_i(\theta)$ with respect to θ satisfies

$$\begin{aligned} V'[\boldsymbol{\zeta}_i(\theta)] &= -\boldsymbol{\zeta}_i^T(\theta)[\gamma_i P_i(\theta) + P_i(\theta)B_i(\theta)R_i^{-1}(\theta)B_i^T(\theta)P_i(\theta)]\boldsymbol{\zeta}_i(\theta) \\ &< 0, \quad \forall\ \boldsymbol{\zeta}_i(\theta) \ne 0 \end{aligned}$$

Hence, $V[\boldsymbol{\zeta}_i(\theta)] < V[\boldsymbol{\zeta}_i(\theta_0)]$, such that $\theta > \theta_0$, from which it follows that

$$\begin{aligned} \|\boldsymbol{u}_i(\theta)\|^2 &= \boldsymbol{u}_i^T(\theta)\boldsymbol{u}_i(\theta) = \boldsymbol{\zeta}_i^T(\theta)P_i(\theta)B_i(\theta)R_i^{-2}(\theta)B_i^T(\theta)P_i(\theta)\boldsymbol{\zeta}_i(\theta) \\ &\le \mathrm{tr}[P_i^{1/2}(\theta)B_i(\theta)R_i^{-2}(\theta)B_i^T(\theta)P_i^{1/2}(\theta)]\boldsymbol{\zeta}_i^T(\theta)P_i(\theta)\boldsymbol{\zeta}_i(\theta) \\ &= \mathrm{tr}[P_i(\theta)B_i(\theta)R_i^{-2}(\theta)B_i^T(\theta)]V[\boldsymbol{\zeta}_i(\theta)] \\ &\le \mathrm{tr}[P_i(\theta)B_i(\theta)R_i^{-2}(\theta)B_i^T(\theta)]V[\boldsymbol{\zeta}_i(\theta_0)] \end{aligned}$$

in which

$$\theta \ge \theta_0$$

$$\lim_{\gamma_i \to 0^+} P_i(\theta) = 0$$

As $\mathcal{X}$ is bounded, there holds

$$\lim_{\gamma_i \to 0^+} \|\boldsymbol{u}_i(\theta)\|^2 = 0$$

and the proof is completed. □

The preceding result can be referred to as semiglobal stabilization of the system described by Eq. (15) with controls of bounded magnitude. The following result further shows that the periodic linear feedback described by Eq. (31) also semiglobally stabilizes the system described by Eq. (15) with controls of bounded energy.

Theorem 3: For any given positive scalar μ_2 and bounded set $\mathcal{X} \subset \mathbf{R}^4$, there exists a scalar $\gamma_i^{**} \in (0, D_i]$ such that the closed-loop system consisting of the periodic system described by Eq. (15) and the control given in Eq. (31) with $\gamma_i \in (0, \gamma_i^{**}]$ is asymptotically stable $\mathcal{X}$ contained in the domain of attraction and such that

$$\|\boldsymbol{u}_i\|_{L_2} \le \mu_2 \quad (34)$$

Proof: Similarly, only Eq. (34) needs to be verified. Let $\boldsymbol{\eta}(\theta) = P_i(\theta)\boldsymbol{\zeta}_i(\theta)$. Then, using Eqs. (26) and (33) gives

$$\begin{aligned} \boldsymbol{\eta}'(\theta) &= P_i'(\theta)\boldsymbol{\zeta}_i(\theta) + P_i(\theta)\boldsymbol{\zeta}_i'(\theta) = \{P_i'(\theta) + P_i(\theta)[A_i(\theta) \\ &\quad + B_i(\theta)K_i(\theta)]\}\boldsymbol{\zeta}_i(\theta) = -[A_i^T(\theta) + \gamma_i I_4]\boldsymbol{\eta}(\theta) \end{aligned}$$

for which the solution is given by (see, for example, [34])

$$\boldsymbol{\eta}(\theta) = \Phi_i^T(\theta_0, \theta) e^{-\gamma_i(\theta-\theta_0)} \boldsymbol{\eta}(\theta_0), \quad \forall\ \theta \geq \theta_0$$

where $\Phi_i(\theta, \theta_0)$ is the state transition matrix for the system described by Eq. (15). With this, and in view of Eq. (30), one can get

$$\begin{aligned}
&\int_{\theta_0}^{\infty} e^{\gamma_i(s-\theta_0)} \boldsymbol{u}_i^T(s) R_i(s) \boldsymbol{u}_i(s)\, \mathrm{d}s \\
&= \int_{\theta_0}^{\infty} e^{\gamma_i(s-\theta_0)} \boldsymbol{\zeta}_i^T(s) P_i(s) B_i(s) R_i^{-1}(s) B_i^T(s) P_i(s) \boldsymbol{\zeta}_i(s)\, \mathrm{d}s \\
&= \int_{\theta_0}^{\infty} e^{\gamma_i(s-\theta_0)} \boldsymbol{\eta}^T(s) B_i(s) R_i^{-1}(s) B_i^T(s) \boldsymbol{\eta}(s)\, \mathrm{d}s \\
&= \boldsymbol{\eta}^T(\theta_0) \left[\int_{\theta_0}^{\infty} e^{-\gamma_i(s-\theta_0)} \Phi_i(\theta_0, s) B_i(s) R_i^{-1}(s) B_i^T(s) \Phi_i^T(\theta_0, s)\, \mathrm{d}s \right] \\
&\times \boldsymbol{\eta}(\theta_0) = \boldsymbol{\eta}^T(\theta_0) W_i(\theta_0) \boldsymbol{\eta}(\theta_0) = \boldsymbol{\zeta}_i^T(\theta_0) P_i(\theta_0) \boldsymbol{\zeta}_i(\theta_0)
\end{aligned}$$

As $R_i(\theta)$ is bounded and nonsingular for all θ, it follows that there exists a constant $\alpha > 0$, such that $R_i(s) \geq \alpha^{-1} I_2$. Consequently,

$$\begin{aligned}
\int_{\theta_0}^{\infty} \|\boldsymbol{u}_i(s)\|^2\, \mathrm{d}s &= \int_{\theta_0}^{\infty} \boldsymbol{u}_i^T(s) \boldsymbol{u}_i(s)\, \mathrm{d}s \\
&\leq \alpha \int_{\theta_0}^{\infty} e^{\gamma_i(s-\theta_0)} \boldsymbol{u}_i^T(s) R_i(s) \boldsymbol{u}_i(s)\, \mathrm{d}s = \alpha \boldsymbol{\zeta}_i^T(\theta_0) P_i(\theta_0) \boldsymbol{\zeta}_i(\theta_0)
\end{aligned}$$

Again, as

$$\lim_{\gamma_i \to 0^+} P_i(\theta) = 0$$

where $\theta \in \mathbf{R}$ and $\mathcal{X}$ is bounded, it follows that

$$\lim_{\gamma_i \to 0^+} \int_{\theta_0}^{\infty} \|\boldsymbol{u}_i(s)\|^2 \mathrm{d}s = 0$$

and the proof is completed. □

Basically, the preceding two theorems mean that the spacecraft rendezvous in the x-z plane can be achieved with the linear periodic controller described by Eq. (31) that has arbitrarily small energy and arbitrarily small magnitude by decreasing the values of γ_i. However, if γ_i is too small, the transient performances of the closed-loop system will be rather poor; say, the settling time is long. In practice, the selection of values of γ_i should be a tradeoff between the settling time and the applicable control energy (and/or the maximal applicable magnitude of the control signals). This tradeoff can be illustrated through the simulation results in Sec. IV.

The PLDE-based approach can also be used to achieve semiglobal stabilization of the out-of-plane motion by controls of bounded magnitude and bounded energy. As $[A_o, B_o(\theta)]$ is ANCBC and NCVE, according to Lemma 2, the following 2π-periodic Riccati differential equation

$$\begin{aligned}
&-P_o'(\theta) = A_o^T(\theta) P_o(\theta) + P_o(\theta) A_o(\theta) + \gamma_o P_o(\theta) \\
&\quad - P_o(\theta) B_o(\theta) R_o^{-1}(\theta) B_o^T(\theta) P_o(\theta)
\end{aligned} \tag{35}$$

where $R_i(\theta)$ is an arbitrary 2π-periodic positive definite matrix and $\gamma_o \in (0, D_o]$, $D_o > 0$, has a unique 2π-periodic positive definite solution $P_o(\theta) = P_o(\theta, \gamma_o)$, such that

$$K_o(\theta) = -R_o^{-1}(\theta) B_o^T(\theta) P_o(\theta) \tag{36}$$

is a stabilizing gain for the linearized out-of-plane motion described by Eq. (18). Moreover,

$$\lim_{\gamma_o \to 0^+} P_o(\theta) = 0, \quad \forall\ \theta \in \mathbf{R} \tag{37}$$

and $P_o(\theta) = W_o^{-1}(\theta)$, with $W_o(\theta)$ being the unique 2π-periodic positive definite solution to the following 2π-periodic PLDE,

$$W_o'(\theta) = A_o(\theta, \gamma_o) W_o(\theta) + W_o(\theta) A_o^T(\theta, \gamma_o) - B_o(\theta) R_o^{-1}(\theta) B_o^T(\theta) \tag{38}$$

in which $A_o(\theta, \gamma_o) = A_o(\theta) + \frac{1}{2}\gamma_o I_2$. The situation in this case can be much more simple if

$$R_o(\theta) = \frac{1}{k^8 \rho^6(\theta)} \tag{39}$$

In fact, in this case, all the coefficient matrices described by Eq. (35) are constant; consequently, the unique 2π-periodic positive definite solution can be solved as

$$P_o(\theta) = \begin{bmatrix} \gamma_o(\gamma_o^2 + 2) & \gamma_o^2 \\ \gamma_o^2 & 2\gamma_o \end{bmatrix} \tag{40}$$

which is independent of θ. Hence, the stabilizing feedback gain can be computed as

$$K_o(\theta) = -k^4 \rho^3(\theta) [\gamma_o^2 \quad 2\gamma_o] \tag{41}$$

In fact, with the periodic state feedback $\boldsymbol{u}_o(\theta) = K_o(\theta) \boldsymbol{\zeta}_o(\theta)$, the closed-loop system matrix $A_o(\theta) + B_o(\theta) K_o(\theta)$ is time invariant and has the eigenvalue set $\{-\gamma_o, -\gamma_o\}$. Finally, it can be readily shown that Theorems 2 and 3 are also true for the out-of-plane motion; the details are omitted for brevity.

B. Algorithm for Rendezvous

In this subsection, an algorithm will be presented for orbital rendezvous by using the PLDE-based approach developed in the previous subsection. First, a periodic generator approach to computing the unique 2π-periodic positive definite solution to the 2π-periodic PLDE described by Eq. (27) will be introduced by adopting the idea found in [37].

Lemma 3: Let $W_i(\theta)$ be the unique 2π-periodic positive definite solution to the 2π-periodic PLDE described by Eq. (27). Then, $W_i(\theta_0)$ satisfies the following discrete-time Lyapunov equation:

$$W_i(\theta_0) = e^{2\pi\gamma_i} G_i(\theta_0) W_i(\theta_0) G_i^T(\theta_0) - Q_i(\theta_0 + 2\pi, \theta_0) \tag{42}$$

where $G_i(\theta_0) = \Phi_i(\theta_0 + 2\pi, \theta_0)$, and

$$Q_i(\theta, \theta_0) = \int_{\theta_0}^{\theta} \Phi_i(\theta, s) B_i(s) R_i^{-1}(s) B_i^T(s) \Phi_i^T(\theta, s) e^{(\theta - s)\gamma_i}\, \mathrm{d}s \tag{43}$$

Proof: For clarity, denote $B_i(s) R_i^{-1}(s) B_i^T(s) = S_i(s)$. Then, it follows from Eq. (30) that

$$\begin{aligned}
W_i(\theta_0 + 2\pi) &= \int_{\theta_0 + 2\pi}^{\infty} \Phi_i(\theta_0 + 2\pi, s) S_i(s) \Phi_i^T(\theta_0 \\
&\quad + 2\pi, s) e^{(\theta_0 + 2\pi - s)\gamma_i}\, \mathrm{d}s = \Phi_i(\theta_0 + 2\pi, \theta_0) \int_{\theta_0 + 2\pi}^{\infty} \Phi_i(\theta_0, \theta_0 \\
&\quad + 2\pi) \Phi_i(\theta_0 + 2\pi, s) S_i(s) \Phi_i^T(\theta_0 + 2\pi, s) \Phi_i^T(\theta_0, \theta_0 \\
&\quad + 2\pi) e^{(\theta_0 + 2\pi - s)\gamma_i}\, \mathrm{d}s \Phi_i^T(\theta_0 + 2\pi, \theta_0) \\
&= G_i(\theta_0) e^{2\pi\gamma_i} \int_{\theta_0 + 2\pi}^{\infty} \Phi_i(\theta_0, s) S_i(s) \Phi_i^T(\theta_0, s) e^{(\theta_0 - s)\gamma_i}\, \mathrm{d}s G_i^T(\theta_0) \\
&= G_i(\theta_0) e^{2\pi\gamma_i} \int_{\theta_0 + 2\pi}^{\theta_0} \Phi_i(\theta_0, s) S_i(s) \Phi_i^T(\theta_0, s) e^{(\theta_0 - s)\gamma_i}\, \mathrm{d}s G_i^T(\theta_0) \\
&+ G_i(\theta_0) \int_{\theta_0}^{\infty} \Phi_i(\theta_0, s) S_i(s) \Phi_i^T(\theta_0, s) e^{(\theta_0 - s)\gamma_i}\, \mathrm{d}s G_i^T(\theta_0) \\
&= G_i(\theta_0) e^{2\pi\gamma_i} W_i(\theta_0) G_i^T(\theta_0) - Q_i(\theta_0 + 2\pi, \theta_0)
\end{aligned}$$

which, in view of $W_i(\theta_0 + 2\pi) = W_i(\theta_0)$, is just Eq. (42). The proof is completed. □

An approach for computing $Q_i(\theta, \theta_0)$ will be established next.

Lemma 4: The matrix $Q_i(\theta, \theta_0)$ satisfies the following Lyapunov differential equation

$$\begin{aligned}
Q_i'(\theta, \theta_0) &= A_i(\theta, \gamma_i) Q_i(\theta, \theta_0) + Q_i(\theta, \theta_0) A_i^T(\theta, \gamma_i) \\
&\quad + B_i(\theta) R_i^{-1}(\theta) B_i^T(\theta)
\end{aligned} \tag{44}$$

with initial condition $Q_i(\theta_0, \theta_0) = 0$. Here, $A_i(\theta, \gamma_i) = A_i(\theta) + \frac{1}{2}\gamma_i I_4$.

Proof: It is clear that $Q_i(\theta_0, \theta_0) = 0$. By definition of $\Phi_i(\theta, \tau)$,

$$\begin{aligned} Q_i'(\theta, \theta_0) &= \Phi_i(\theta, \theta) S_i(\theta) \Phi_i^T(\theta, \theta) \\ &+ \int_{\theta_0}^{\theta} \left[\frac{\mathrm{d}}{\mathrm{d}\theta} \Phi_i(\theta, s)\right] S_i(s) \Phi_i^T(\theta, s) e^{(\theta - s)\gamma_i} \, \mathrm{d}s \\ &+ \int_{\theta_0}^{\theta} \Phi_i(\theta, s) S_i(s) \left[\frac{\mathrm{d}}{\mathrm{d}\theta} \Phi_i(\theta, s)\right]^T e^{(\theta - s)\gamma_i} \, \mathrm{d}s \\ &+ \int_{\theta_0}^{\theta} \Phi_i(\theta, s) S_i(s) \Phi_i^T(\theta, s) \gamma_i e^{(\theta - s)\gamma_i} \, \mathrm{d}s = S_i(\theta) \\ &+ A_i(\theta) Q_i(\theta, \theta_0) + Q_i(\theta, \theta_0) A_i^T(\theta) + \gamma_i Q_i(\theta, \theta_0) \end{aligned}$$

The proof is finished. □

The idea of obtaining the unique 2π-periodic positive definite solution to the PLDE described by Eq. (27) can be stated as follows. If $Q_i(\theta_0 + 2\pi, \theta_0)$ is obtained by solving the linear differential equation described by Eq. (44) according to Lemma 4, $W_i(\theta_0)$ can be computed by solving the Lyapunov equation described by Eq. (42) according to Lemma 3. Consequently, the 2π-periodic positive definite solution $W_i(\theta)$ can be obtained by solving the 2π-periodic PLDE described by Eq. (27) with initial condition $W_i(\theta_0)$; namely, such initial condition $W_i(\theta_0)$ generates the unique 2π-periodic positive definite solution to the PLDE described by Eq. (27).

To solve the linear differential equations described by Eqs. (27) and (44), the following approach is developed.

Definition 6: Let $P = [p_{ij}] \in \mathbf{R}^{n \times n}$ be a symmetric matrix. Then, the symmetric stretching function $\mathrm{sv}(\cdot)$: $\mathbf{R}^{n \times n} \to \mathbf{R}^{(1/2)n(n+1)}$ is defined as

$$\mathrm{sv}\,(P) = [\, p_{11} \quad p_{12} \quad p_{22} \quad p_{13} \quad p_{23} \quad p_{33} \quad \cdots \quad p_{1n} \quad p_{2n} \quad \cdots \quad p_{nn} \,]^T$$

Moreover, the inverse symmetric stretching function $\mathrm{sv}^{-1}(\cdot)$: $\mathbf{R}^{(1/2)n(n+1)} \to \mathbf{R}^{n \times n}$ can be defined in an obvious way.

Clearly, $\mathrm{sv}(\cdot)$ is a linear operator. For any given matrix $A \in \mathbf{R}^{n \times n}$, there exists a unique matrix

$$T(A) \in \mathbf{R}^{[(1/2)n(n+1)] \times [(1/2)n(n+1)]}$$

such that

$$\mathrm{sv}\,(AP + PA^T) = T(A)\mathrm{sv}(P) \tag{45}$$

For example, if $A = A_i(\theta) \in \mathbf{R}^{4 \times 4}$ is given by Eq. (16), then

$$T[A_i(\theta)] = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & \frac{3}{\rho} & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{\rho} & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{3}{\rho} & -2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{\rho} & -4 & 0 \end{bmatrix} \tag{46}$$

By using Eq. (45), taking $\mathrm{sv}(\cdot)$ on both sides of Eq. (44) gives

$$\boldsymbol{\phi}_i'(\theta) = \{T[A_i(\theta)] + \gamma_i I_{10}\}\boldsymbol{\phi}_i(\theta) + \mathrm{sv}[B_i(\theta) R_i^{-1}(\theta) B_i^T(\theta)] \tag{47}$$

where $\boldsymbol{\phi}_i(\theta) = \mathrm{sv}[Q_i(\theta, \theta_0)]$ and $\boldsymbol{\phi}_i(\theta_0) = 0$. Similarly, Eq. (27) can be transformed into

$$\boldsymbol{\varphi}_i'(\theta) = \{T[A_i(\theta)] + \gamma_i I_{10}\}\boldsymbol{\varphi}_i(\theta) - \mathrm{sv}[B_i(\theta) R_i^{-1}(\theta) B_i^T(\theta)] \tag{48}$$

in which $\boldsymbol{\varphi}_i(\theta) = \mathrm{sv}[W_i(\theta)]$ and $\boldsymbol{\varphi}_i(\theta_0) = \mathrm{sv}[W_i(\theta_0)]$. Then, the following algorithm for computing the stabilizing feedback gain $K_i(\theta)$, as defined in Eq. (28), can be presented.

Algorithm 1: [For any given $\theta_0 \in [0, 2\pi)$, compute $K_i(\theta)$, $\theta \geq \theta_0$, as defined in Eq. (28)].

1) Solve the linear differential equation described by Eq. (47) to get

$$Q_i(\theta_0 + 2\pi, \theta_0) = \mathrm{sv}^{-1}[\boldsymbol{\phi}_i(\theta_0 + 2\pi)]$$

2) Solve the following differential equation to get $J(\theta_0)$

$$J'(\theta) = \frac{1}{\rho^2(\theta)}, \qquad J(0) = 0 \tag{49}$$

3) Use $J(\theta_0)$ to compute $G_i(\theta_0) = I_4 + N_i(\theta_0) F_i^{-1}(\theta_0)$, in which $F_i(\theta_0)$ is defined in Eq. (22) and

$$N_i(\theta_0) = \begin{bmatrix} 0 & 0 & 0 & 3\rho^2(\theta_0) \\ 0 & 0 & 0 & -3e\rho(\theta_0)\sin(\theta_0) \\ 0 & 0 & 0 & -6e\rho(\theta_0)\sin(\theta_0) \\ 0 & 0 & 0 & -3e[\cos(\theta_0) + e\cos(2\theta_0)] \end{bmatrix} k^2 T$$

4) Solve the discrete-time Lyapunov equation described by Eq. (42) to get $W_i(\theta_0)$ by using $Q_i(\theta_0 + 2\pi, \theta_0)$ and $G_i(\theta_0)$.

5) Use $W_i(\theta_0)$ to solve the 2π-periodic PLDE described by Eq. (48) to get $\boldsymbol{\varphi}_i(\theta)$, $\theta \geq \theta_0$. Let $W_i(\theta) = \mathrm{sv}^{-1}[\boldsymbol{\varphi}_i(\theta)]$. Then, the stabilizing feedback gain is given by

$$K_i(\theta) = -R_i^{-1}(\theta) B_i^T(\theta) W_i^{-1}(\theta), \qquad \theta \geq \theta_0 \tag{50}$$

Some remarks for the preceding algorithm are in order.

Remark 2: The computation of $G_i(\theta_0)$ in the third step is based on Eqs. (A4) and (A5) and Lemma 5 in the Appendix. Moreover, $J(\theta_0)$ can be computed alternatively as follows. According to Eq. (A3), it can be observed that $J(\theta_0) = k^2 t_0$, where t_0 can be computed by Kepler's equation, described by Eq. (6) as

$$t_0 = \sqrt{\frac{a^3}{\mu}}(E_0 - e\sin(E_0))$$

in which $E_0 \in [0, 2\pi)$ and $\theta_0 \in [0, 2\pi)$ satisfy Eq. (5).

Remark 3: Notice that $Q_i(\theta_0 + 2\pi, \theta_0)$, $J(\theta_0)$, $G_i(\theta_0)$, and $W_i(\theta_0)$ are only functions of the initial true anomaly θ_0. If $\theta_0 = 0$, namely, the target spacecraft is at perigee when $t_0 = 0$, then $J(\theta_0) = 0$, $J(2\pi) = k^2 T$, and

$$G_i(\theta_0) = \begin{bmatrix} 1 & -2 - e & 0 & 3(1+e)^2 k^2 T \\ 0 & 0 & 1 + e & 2 \\ 0 & 0 & 2 + e & 3 \\ 0 & 1 + e & 0 & -3e(1+e)k^2 T \end{bmatrix}$$

Remark 4: In general, k is relatively small; hence, the elements in $B_i(\theta)$ are very large compared with the elements in $A_i(\theta)$. To ensure the computation of the solution to the differential equations described by Eqs. (47) and (48) to possess high accuracy, one can specifically choose

$$R_i(\theta) = \frac{1}{k^8 \rho^6(\theta)} I_2 \tag{51}$$

Consequently, $B_i(\theta) R_i^{-1}(\theta) B_i^T(\theta)$ is a constant matrix, and

$$\mathrm{sv}\,[B_i(\theta) R_i^{-1}(\theta) B_i^T(\theta)] = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1]^T \tag{52}$$

IV. Simulation Results

In this section, a numerical simulation will be used to demonstrate the effectiveness of the proposed approach to elliptical orbital rendezvous control. Different from the controller design, our simulation will be carried out directly on the nonlinear plant described by Eq. (2). To this end, the state transformation described by Eq. (13) should first be applied on the nonlinear plant described by Eq. (2); namely,

Table 1 The orbital parameters of the target spacecraft

Parameters	Symbol	Values
Semimajor axis	a	2.4616×10^7 m
Eccentricity	e	0.73074
Angular momentum	h	6.762×10^{10} m^2/s
Constant k	k	$2.267 \times 10^{-2}/\mathrm{s}^{1/2}$
Period	T	38,436 s
Gravity constant	μ	3.986×10^{14} m^3/s^2

$$\boldsymbol{\zeta}(\theta) \triangleq \begin{bmatrix} \boldsymbol{\zeta}_i(\theta) \\ \boldsymbol{\zeta}_o(\theta) \end{bmatrix} = L(\theta) \begin{bmatrix} \boldsymbol{\xi}_i(t) \\ \boldsymbol{\xi}_o(t) \end{bmatrix} \triangleq L(\theta)\boldsymbol{\xi}(t) \tag{53}$$

where $L(\theta) = \mathrm{diag}\{L_i(\theta), L_o(\theta)\}$, with $L_i(\theta)$ and $L_o(\theta)$ given by Eqs. (14) and (17), respectively. Consequently, according to the results obtained in Sec. III, the overall controller is given by

$$\boldsymbol{u}(\theta) = \begin{bmatrix} \boldsymbol{u}_i(\theta) \\ \boldsymbol{u}_o(\theta) \end{bmatrix} = \begin{bmatrix} K_i(\theta)\boldsymbol{\zeta}_i(\theta) \\ K_o(\theta)\boldsymbol{\zeta}_o(\theta) \end{bmatrix} = \begin{bmatrix} K_i(\theta) & 0 \\ 0 & K_o(\theta) \end{bmatrix} \boldsymbol{\zeta}(\theta) \tag{54}$$

in which $K_i(\theta)$ and $K_o(\theta)$ are defined in Eqs. (28) and (41), respectively.

Suppose that the target spacecraft is in the geostationary transfer orbit, which is a temporary orbit to inject a satellite into the geostationary Earth orbit (see, for example, [20]). The orbital parameters are as follows: the semimajor axis $a = 24616$ km, the eccentricity $e = 0.73074$, and the period $T = 38436$ s. For clarity, the main parameters are listed in Table 1.

Assume that at the initial time, the true anomaly is $\theta_0 = 0.1\pi$. Take the in-plane motion into consideration first. By choosing $\gamma_i = 1.1$, according to the steps in the Algorithm, one can get in the first step,

$$Q_i(\theta_0 + 2\pi, \theta_0) = \begin{bmatrix} 5.380 & -0.7156 & -1.437 & -2.113 \\ -0.7156 & 0.0952 & 0.1912 & 0.2810 \\ -1.437 & 0.1912 & 0.3839 & 0.5643 \\ -2.113 & 0.2810 & 0.5643 & 0.8296 \end{bmatrix} \times 10^8 \tag{55}$$

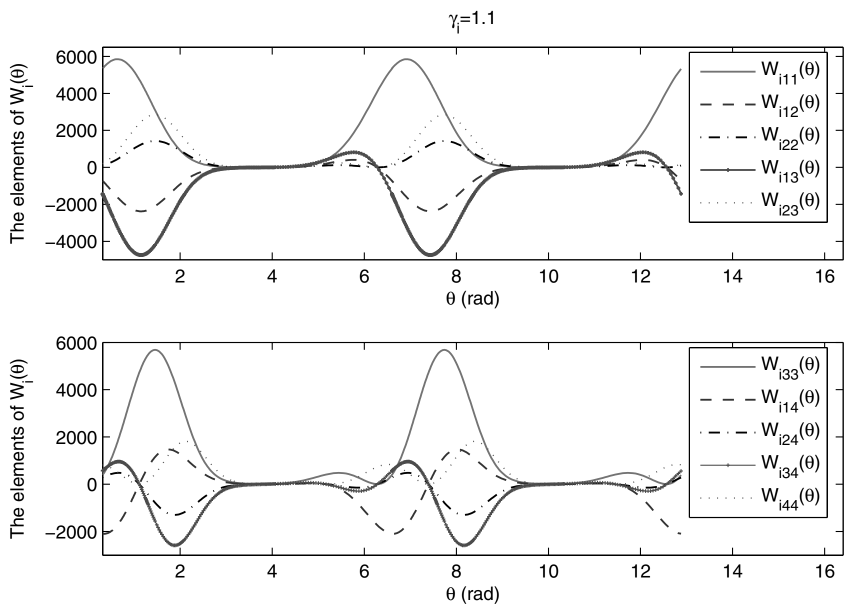

Fig. 2 Elements in $W_i(\theta) = [W_{ijk}(\theta)]$; $j, k = 1, 2, 3, 4$.

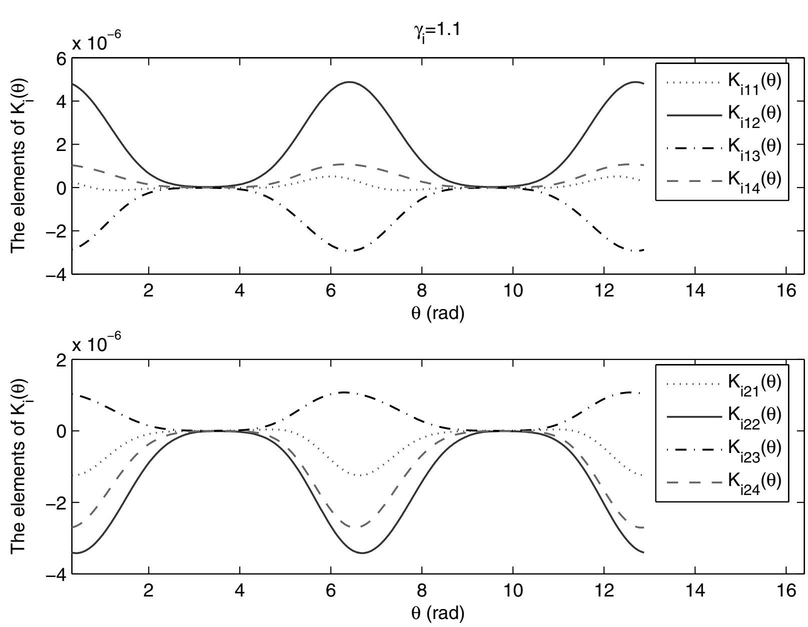

Fig. 3 Elements in $K_i(\theta) = [K_{ijk}(\theta)]$; $j = 1, 2$ and $k = 1, 2, 3, 4$.

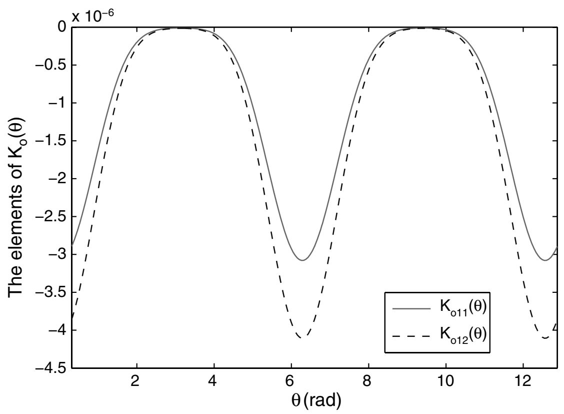

Fig. 4 Elements in $K_o(\theta) = [K_{o11}(\theta) K_{o12}(\theta)]$.

In the second step, one obtains $J(\theta_0) = 0.106355$ and $J(\theta_0 + 2\pi) = 19.7507$. Consequently, in the third step, one can compute

$$G_i(\theta_0) = \begin{bmatrix} 0.001 & 1.687 & -1.049 & 0.1398 \\ 0 & -0.2238 & 0.1398 & -0.01863 \\ 0 & -0.4496 & 0.2806 & -0.03725 \\ 0 & -0.6619 & 0.4117 & -0.05384 \end{bmatrix} \times 10^3 \quad (56)$$

Then, by using Eqs. (55) and (56), the unique positive definite solution $W_i(\theta_0)$ to the discrete-time Lyapunov equation, described by Eq. (42), can be obtained as

$$W_i(\theta_0) = \begin{bmatrix} 5.341 & -0.7209 & -1.434 & -2.100 \\ -0.7209 & 0.0984 & 0.1951 & 0.2827 \\ -1.434 & 0.1951 & 0.3876 & 0.5630 \\ -2.100 & 0.2827 & 0.5630 & 0.8271 \end{bmatrix} \times 10^3$$

Hence, the periodic matrix $W_i(\theta)$, $\theta \geq \theta_0$, can be computed by solving the linear differential equation described by Eq. (48), from which the periodic feedback gain $K_i(\theta)$ can be calculated according to Eq. (50). The results are recorded in Fig. 2. It follows that both $W_i(\theta)$ and $K_i(\theta)$ are 2π periodic. Similarly, for the out-of-plane motion, with the choice of $\gamma_o = 1.5$, the feedback gain $K_o(\theta)$, $\theta \geq \theta_0$, can be computed according to Eq. (41). The results are plotted in Figs. 3 and 4. It is very interesting to notice that the absolute values of all the elements in $K_i(\theta)$ and $K_o(\theta)$ achieve their minima at apogee. This is reasonable, as the target spacecraft at apogee has the smallest orbital rate so that the controls required by the chaser so as to follow the target also achieve their minima.

For simulation purposes, choose the initial condition in the target-orbital coordinate system as

$$\boldsymbol{\xi}(\theta_0) = [10{,}000 \quad -10{,}000 \quad 3 \quad 3 \quad 10{,}000 \quad -3]^T \qquad (57)$$

namely, all the distances between the target and the chaser spacecraft in all three directions, x, y, and z, are 10,000 m. According to Eq. (53), the initial condition in the $\boldsymbol{\zeta}$-coordinate system is

$$\boldsymbol{\zeta}(\theta_0) = L(\theta_0)\boldsymbol{\xi}(\theta_0) = [1.695 \quad -1.695 \quad 0.1186 \quad 0.5702 \quad 1.695 \quad -0.5702]^T \times 10^4$$

By simulation, the state trajectories and control signals of the closed-loop system consisting of the transformed nonlinear T–H equations and the linear controller described by Eq. (54) are, respectively, recorded in Figs. 5 and 6. It follows that the closed-loop system is asymptotically stable. Actually, the rendezvous mission is accomplished at about $\theta_f = 5.2$. The settling time can be computed as

$$T_s = [(\theta_f - \theta_0)/2\pi]T = 2.989 \times 10^4 \text{ s}$$

Next, the same initial condition will be used to show that both the magnitude and energy of the control signals (i.e., the thrust accelerations) can be made as small as desired by choosing the parameters γ_i and γ_o in the controllers, for example, to verify

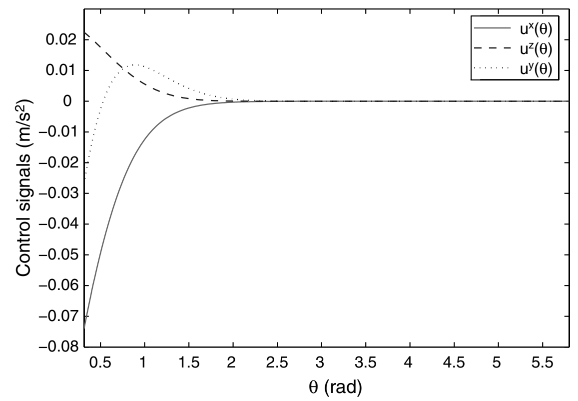

Fig. 6 Control signals with $\gamma_i = 1.1$ and $\gamma_o = 1.5$.

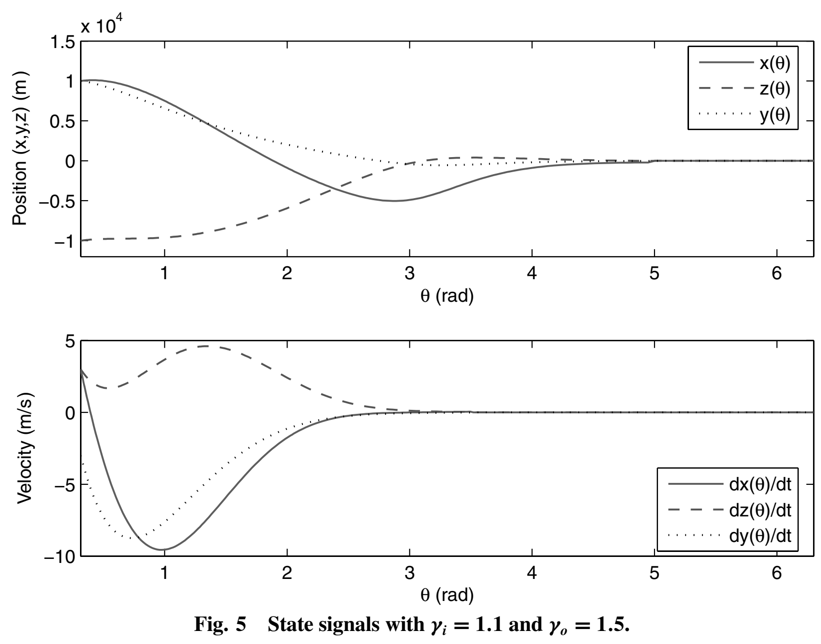

Fig. 5 State signals with $\gamma_i = 1.1$ and $\gamma_o = 1.5$.

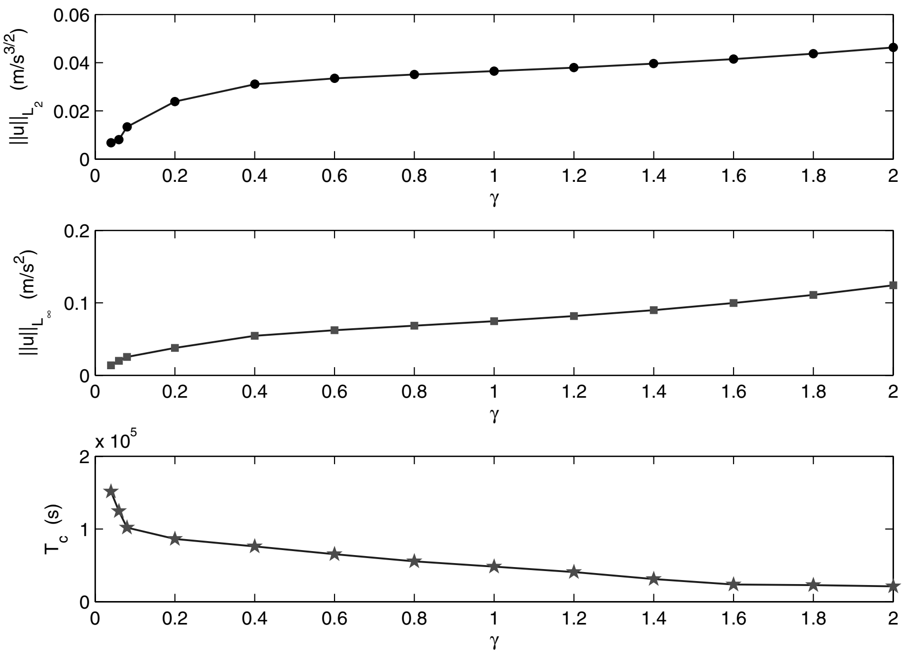

Fig. 7 $\|u\|_{L_\infty}$, $\|u\|_{L_2}$, **and the settling time** T_s **with different values of** γ.

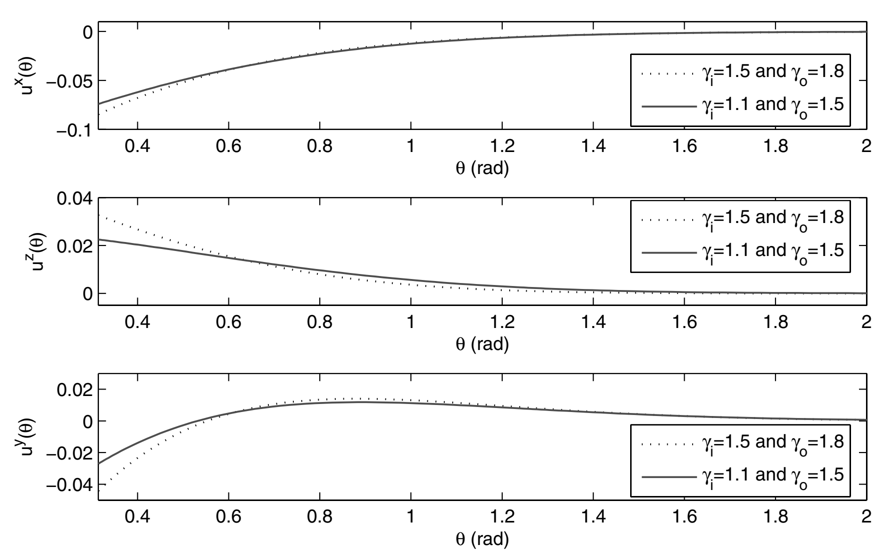

Fig. 8 Control signals with different values of γ_i **and** γ_o.

Theorems 2 and 3. For simplicity, assume that $\gamma_i = \gamma_o = \gamma$. For different values of γ, the values of $\|\boldsymbol{u}\|_{L_\infty}$ and $\|\boldsymbol{u}\|_{L_2}$ are recorded in Fig. 7, which clearly validates the desired statements. The settling time T_s for different values of γ is also recorded in Fig. 7, from which it can be observed that T_s increases as γ decreases. Hence, the proper selection of γ should be a tradeoff between the settling time and the control energy (magnitude) that the thrusts can supply.

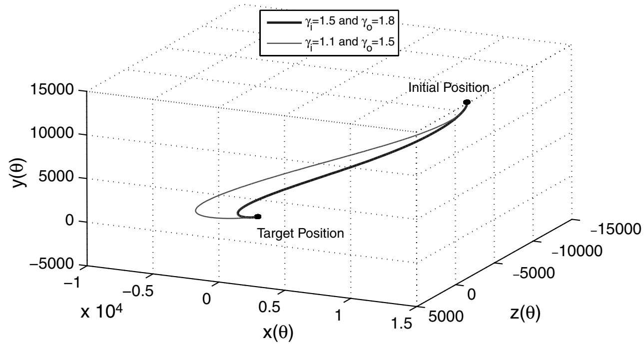

Fig. 9 Trajectories with different values of γ_i **and** γ_o.

Suppose that the maximal accelerations supplied by the thrusts in the three directions satisfy, respectively, $|u^x(\theta)| \leq 0.1$, $|u^z(\theta)| \leq 0.1$, and $|u^y(\theta)| \leq 0.05$. The closed-loop system is simulated for different values of γ_i and γ_o, and it is found that these constraints can be satisfied if $\gamma_i \leq 1.82$ and $\gamma_o \leq 2.01$. For two pairs of different values of γ_i and γ_o, $(\gamma_i, \gamma_o) = (1.1, 1.5)$ and $(\gamma_i, \gamma_o) = (1.5, 1.8)$, the control signals and state trajectories of the closed-loop system are shown in Figs. 8 and 9, respectively. These figures again confirm that larger values for γ_i and γ_o lead to better transient performances at the cost of larger control energy and magnitude of the control signals.

V. Conclusions

This paper has studied the spacecraft rendezvous problem with the target spacecraft in arbitrary eccentric orbits. A novel PLDE-based approach was proposed to solve the problem. With the identification of the Jordan canonical form of the T–H equations, it was clarified that the T–H equations are both asymptotically null controllable with bounded controls and NCVE, which completes the existing results. Based on these controllability properties, it was proven in the paper that the proposed linear periodic controller achieves semiglobal stabilization of the relative motion with magnitude and/or energy-bounded controls. In other words, the rendezvous mission can be accomplished with arbitrarily small control in magnitude and/or energy by using this class of feedback controls. One of the main advantages of the proposed approach over the existing one is that the controller is easy to implement, as only a linear Lyapunov differential equation is required to be solved online. A periodic generator approach was also presented to solve such linear equations. A numerical example was developed to show the effectiveness of the proposed methods.

Appendix

I. Proof of Theorem 1

The following simple lemma will be established first.

Lemma 5: Let $J(\theta)$ be as defined in Eq. (21). Then, for any $\theta_0 \in [0, 2\pi)$,

$$\int_{\theta_0}^{\theta_0+2\pi} \frac{1}{\rho^2(\tau)}\,\mathrm{d}\tau = k^2 T \tag{A1}$$

Proof: It follows from Eq. (4) that $\omega = \mathrm{d}\theta/\mathrm{d}t = k^2\rho^2$; namely,

$$k^2\mathrm{d}t = \frac{1}{\rho^2}\mathrm{d}\theta \tag{A2}$$

Let the true anomaly of the target spacecraft at t_0 be θ_0. Taking integration on both sides of Eq. (A2) gives

$$\int_{\theta_0}^{\theta(t)} \frac{1}{\rho^2(\tau)}\,\mathrm{d}\tau = k^2(t - t_0) \tag{A3}$$

Hence, if $\theta(t) = \theta_0 + 2\pi$, then $t = T + t_0$. The proof is completed. □

According to Eq. (23), it can be computed that

$$G_i(\theta_0) \triangleq \Phi_i(\theta_0 + 2\pi, \theta_0) = F_i(\theta_0 + 2\pi)F_i^{-1}(\theta_0) = [F_i(\theta_0) + N_i(\theta_0)]F_i^{-1}(\theta_0) = I_4 + N_i(\theta_0)F_i^{-1}(\theta_0) \tag{A4}$$

where $N_i(\theta_0) \triangleq F_i(\theta_0 + 2\pi) - F_i(\theta_0)$ can be computed directly as

$$N_i(\theta_0) = \begin{bmatrix} 0 & 0 & 0 & 3\rho^2(\theta_0) \\ 0 & 0 & 0 & -3e\rho(\theta_0)\sin(\theta_0) \\ 0 & 0 & 0 & -6e\rho(\theta_0)\sin(\theta_0) \\ 0 & 0 & 0 & -3e(\cos(\theta_0) + e\cos(2\theta_0)) \end{bmatrix} \int_{\theta_0}^{\theta_0+2\pi} \frac{\mathrm{d}\tau}{\rho^2(\tau)} \tag{A5}$$

Direct manipulation results in

$$F_i^{-1}(\theta_0)N_i(\theta_0) = \begin{bmatrix} 0 & 0 & 0 & 3\int_{\theta_0}^{\theta_0+2\pi}\frac{\mathrm{d}\tau}{\rho^2(\tau)} \\ 0 & 0 & 0 & -3e\int_{\theta_0}^{\theta_0+2\pi}\frac{\mathrm{d}\tau}{\rho^2(\tau)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = V_i(\theta_0)J_1(\theta_0)V_i^{-1}(\theta_0) \tag{A6}$$

in which the matrices $V_i(\theta_0)$ and $J_1(\theta_0)$ are, respectively, given by

$$V_i(\theta_0) = \begin{bmatrix} 3\int_{\theta_0}^{\theta_0+2\pi}\frac{\mathrm{d}\tau}{\rho^2(\tau)} & 1 & 1 & 1 \\ -3e\int_{\theta_0}^{\theta_0+2\pi}\frac{\mathrm{d}\tau}{\rho^2(\tau)} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$J_1(\theta_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

According to Eq. (A1),

$$\det[V_i(\theta_0)] = -3e\int_{\theta_0}^{\theta_0+2\pi}\frac{\mathrm{d}\tau}{\rho^2(\tau)} = -3ek^2T \neq 0, \quad \forall\ \theta_0 \in \mathbf{R}$$

Therefore, the relation described by Eq. (A6) is true for all $\theta_0 \in \mathbf{R}$. Inserting Eq. (A6) into Eq. (A4) gives

$$\begin{aligned} G_i(\theta_0) &= F_i(\theta_0)[I_4 + F_i^{-1}(\theta_0)N_i(\theta_0)]F_i^{-1}(\theta_0) \\ &= F_i(\theta_0)[I_4 + V_i(\theta_0)J_1(\theta_0)V_i^{-1}(\theta_0)]F_i^{-1}(\theta_0) \\ &= F_i(\theta_0)V_i(\theta_0)[I_4 + J_1(\theta_0)]V_i^{-1}(\theta_0)F_i^{-1}(\theta_0) \\ &= F_i(\theta_0)V_i(\theta_0)J_i(\theta_0)[F_i(\theta_0)V_i(\theta_0)]^{-1} \\ &= U_i(\theta_0)J_i(\theta_0)U_i^{-1}(\theta_0) \end{aligned} \tag{A7}$$

in which $U_i(\theta_0) = F_i(\theta_0)V_i(\theta_0)$, and

$$J_i(\theta_0) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is easy to verify that $J_i(\theta_0) = e^{J_i}$. Hence, it follows from Eq. (A7) that

$$G_i(\theta_0) = U_i(\theta_0)e^{J_i}U_i^{-1}(\theta_0) \triangleq e^{2\pi S_i(\theta_0)} \tag{A8}$$

where

$$S_i(\theta_0) = (1/2\pi)U_i(\theta_0)J_iU_i^{-1}(\theta_0)$$

Notice that $(1/2\pi)J_i = V_iJ_iV_i^{-1}$, in which

$$V_i = \begin{bmatrix} \frac{1}{2\pi} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Hence, there holds

$$S_i(\theta_0) = U_i(\theta_0)V_iJ_i[U_i(\theta_0)V_i]^{-1} \tag{A9}$$

Now, let

$$H_i(\theta) = \Phi_i(\theta, \theta_0)e^{-S_i(\theta_0)\theta}U_i(\theta_0)V_i$$

such that $\theta \in \mathbf{R}$. Clearly, $H_i(\theta)$ is nonsingular for all θ. According to the definition of the state transition matrix $\Phi_i(\theta, \vartheta)$ and Eq. (A8), it can be verified that

$$\begin{aligned} H_i(\theta + 2\pi) &= \Phi_i(\theta + 2\pi, \theta_0)e^{-S_i(\theta_0)(\theta+2\pi)}U_i(\theta_0)V_i \\ &= \Phi_i(\theta + 2\pi, \theta_0)e^{-2\pi S_i(\theta_0)}e^{-S(\theta_0)\theta}U_i(\theta_0)V_i \\ &= \Phi_i(\theta + 2\pi, \theta_0 + 2\pi)\Phi_i(\theta_0 + 2\pi, \theta_0)G_i^{-1}(\theta_0)e^{-S_i(\theta_0)\theta}U_i(\theta_0)V_i \\ &= \Phi_i(\theta, \theta_0)G_i(\theta_0)G^{-1}(\theta_0)e^{-S_i(\theta_0)\theta}U_i(\theta_0)V_i = H_i(\theta) \end{aligned}$$

Hence, $H_i(\theta)$ is a 2π-periodic Lyapunov transformation matrix. Let $\boldsymbol{\zeta}_i(\theta) = H_i(\theta)\boldsymbol{\zeta}_{it}(\theta)$. Then, the system described by Eq. (15) is transformed into

$$\boldsymbol{\zeta}'_{it}(\theta) = A_{iJ}(\theta)\boldsymbol{\zeta}_{it}(\theta) + B_{iJ}(\theta)\boldsymbol{v}_i(\theta)$$

in which $B_{iJ}(\theta) = H_i^{-1}(\theta)B_i(\theta)$. In what follows, matrix $A_{iJ}(\theta)$ will be simplified. By using Eq. (A9), it can be computed that

$$\begin{aligned}A_{iJ} &= H_i^{-1}(\theta)[A_i(\theta)H_i(\theta) - H_i'(\theta)]\\ &= H_i^{-1}(\theta)\left[A_i(\theta)H_i(\theta) - \frac{\mathrm{d}}{\mathrm{d}\theta}\Phi_i(\theta,\theta_0)e^{-S_i(\theta_0)\theta}U_i(\theta_0)V_i\right]\\ &= H_i^{-1}(\theta)[A_i(\theta)H_i(\theta) - A_i(\theta)\Phi_i(\theta,\theta_0)e^{-S_i(\theta_0)\theta}U_i(\theta_0)V_i]\\ &+ H_i^{-1}(\theta)\Phi_i(\theta,\theta_0)e^{-S_i(\theta_0)\theta}S_i(\theta_0)U_i(\theta_0)V_i\\ &= H_i^{-1}(\theta)\{A_i(\theta)H_i(\theta) - A_i(\theta)H_i(\theta)\\ &+ \Phi_i(\theta,\theta_0)e^{-S_i(\theta_0)\theta}[U_i(\theta_0)V_i][U_i(\theta_0)V_i]^{-1}S_i(\theta_0)U_i(\theta_0)V_i\}\\ &= [U_i(\theta_0)V_i]^{-1}S_i(\theta_0)U_i(\theta_0)V_i = J_i\end{aligned}$$

From Definition 3, the Jordan canonical form of the system described by Eq. (15) is J_i. The proof is completed.

II. Proof of Lemma 2

Before proving this lemma, a useful result regarding the periodic Lyapunov differential equation is introduced.

Lemma 6 [38]: The ω-periodic linear system described by Eq. (20), where $u(t) = 0$ is asymptotically stable if, and only if, the following Lyapunov differential equation,

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + C^T(t)C(t)$$

has a unique positive definite ω-periodic solution $P(t)$, where $C(t)$ is a continuous ω-periodic matrix, and such that $[A(t), C(t)]$ is observable.

Proof of item 1: Multiplying both sides of Eq. (26) by $P_i^{-1}(\theta)$ and denoting $W_i(\theta) = P_i^{-1}(\theta)$ gives

$$-W_i'(\theta) = \mathcal{A}_i^T(\theta)W_i(\theta) + W_i(\theta)\mathcal{A}_i(\theta) + C_i^T(\theta)C_i(\theta) \quad \text{(A10)}$$

where

$$\mathcal{A}_i(\theta,\gamma_i) = -A_i^T(\theta) - \tfrac{1}{2}\gamma_i I_4, \qquad C(\theta) = R_i^{-1/2}(\theta)B_i^T(\theta) \quad \text{(A11)}$$

Obviously, the controllability of $[A_i(\theta), B_i(\theta)]$ implies the controllability of $[A_i(\theta), B_i(\theta)R^{-1/2}(\theta)]$, which further implies the controllability of $[A_i(\theta) + \frac{1}{2}\gamma_i I_4, B_i(\theta)R_i^{-1/2}(\theta)]$ or, equivalently, the observability of $[\mathcal{A}_i(t), C_i(\theta)]$.

Let $\Phi(\theta,\vartheta)$ be the state transition matrix of the periodic linear system described by Eq. (15). Then, a simple manipulation shows that the state transition matrix of the following system

$$\boldsymbol{\eta}'(\theta) = \mathcal{A}_i(\theta,\gamma_i)\boldsymbol{\eta}(\theta) = [-A_i^T(\theta) - \tfrac{1}{2}\gamma_i I_4]\boldsymbol{\eta}(\theta) \quad \text{(A12)}$$

is given by

$$\Phi_1(\theta,\vartheta) = \Phi^T(\vartheta,\theta)\exp[-\tfrac{1}{2}\gamma_i(\theta-\vartheta)]$$

With this, and in view of Lemma 1, it can be verified that

$$\begin{aligned}\mathcal{C}[\mathcal{A}_i(\theta,\gamma_i)] &= \lambda[\Phi_1(2\pi,0)] = \lambda[\Phi^T(0,2\pi)e^{-\pi\gamma_i}]\\ &= \lambda[\Phi(0,2\pi)e^{-\pi\gamma_i}] = \lambda[\Phi^{-1}(2\pi,0)e^{-\pi\gamma_i}]\\ &= \left\{\frac{e^{-\pi\gamma_i}}{s}: s\in\lambda[\Phi(2\pi,0)]\right\} = \left\{\frac{e^{-\pi\gamma_i}}{s}: s\in\mathcal{C}[A_i(\theta)]\right\}\\ &= \{e^{-\pi\gamma_i}, e^{-\pi\gamma_i}, e^{-\pi\gamma_i}, e^{-\pi\gamma_i}\} \quad \text{(A13)}\end{aligned}$$

where the relation $\mathcal{C}[A_i(\theta)] = \{1,1,1,1\}$ has been used. Therefore, for any $\gamma_i > 0$, the linear periodic system described by Eq. (A12) is asymptotically stable. With this fact and the observability of $[\mathcal{A}_i(\theta), C_i(\theta)]$, it can be deduced from Lemma 6 that the periodic Lyapunov equation described by Eq. (A10) has a unique positive definite solution $W_i(\theta)$. The proof of this item is finished by noting that Eq. (A10) is equivalent to Eq. (27).

Proof of item 2: Under the state feedback $\boldsymbol{u}_i(\theta) = K_i(\theta)\boldsymbol{\zeta}_i(\theta)$, the closed-loop system reads

$$\boldsymbol{\zeta}_i'(\theta) = [A_i(\theta) + B_i(t)K_i(\theta)]\boldsymbol{\zeta}_i(\theta) \triangleq A_{ci}(\theta)\boldsymbol{\zeta}_i(\theta) \quad \text{(A14)}$$

Simple manipulation shows that $A_{ci}(\theta)$ satisfies the PLDE,

$$\begin{aligned}A_{ci}^T(\theta)P_i(\theta) + P_i(\theta)A_{ci}(\theta) + P_i'(\theta) &= -\gamma_i P_i(\theta)\\ &- P_i(\theta)B_i(\theta)R_i^{-1}(\theta)B_i^T(\theta)P_i(\theta)\end{aligned}$$

The stability of the closed-loop system described by Eq. (A14) then follows from Lemma 6 directly.

Proof of item 3: Taking the partial derivative on both sides of Eq. (26) with respect to γ_i gives

$$\begin{aligned}-\frac{\partial P_i'(\theta)}{\partial\gamma_i} &= \left[A_{ci}(\theta) + \frac{1}{2}\gamma_i I_4\right]^T\frac{\partial P_i(\theta)}{\partial\gamma_i}\\ &+ \frac{\partial P_i(\theta)}{\partial\gamma_i}\left[A_{ci}(\theta) + \frac{1}{2}\gamma_i I_4\right] + P_i(\theta) \quad \text{(A15)}\end{aligned}$$

in which $A_{ci}(\theta)$ is defined in Eq. (A14). On the other hand, it is noted that the periodic Riccati differential equation described by Eq. (26) can be rewritten as

$$\begin{aligned}-P_i'(\theta) &= [A_i(\theta) + \tfrac{1}{2}\gamma_i I_4]^T P_i(\theta) + P_i(\theta)[A_i(\theta) + \tfrac{1}{2}\gamma_i I_4\\ &- B_i(\theta)R_i^{-1}(\theta)B_i^T(\theta)P_i(\theta)] = [A_i(\theta) + \tfrac{1}{2}\gamma_i I_4]^T P_i(\theta)\\ &+ P_i(\theta)[A_{ci}(\theta) + \tfrac{1}{2}\gamma_i I_4] = P_i(\theta)[A_{ci}(\theta) + \tfrac{1}{2}\gamma_i I_4]\\ &- \mathcal{A}_i(\theta,\gamma_i)P_i(\theta) \quad \text{(A16)}\end{aligned}$$

where $\mathcal{A}_i(\theta,\gamma_i)$ is defined in Eq. (A11). Consider the Lyapunov transformation $P_i^{-1}(\theta)\boldsymbol{\zeta}(\theta) = \boldsymbol{\chi}(\theta)$. Then, it follows from Eq. (A16) that the linear periodic system described by Eq. (A12) is transformed into

$$\begin{aligned}\boldsymbol{\chi}'(\theta) &= [P_i^{-1}(\theta)]'\boldsymbol{\eta}(\theta) + P_i^{-1}(\theta)\boldsymbol{\eta}'(\theta)\\ &= -P_i^{-1}(\theta)P_i'(\theta)P_i^{-1}(\theta)\boldsymbol{\eta}(\theta) + P_i^{-1}(\theta)\mathcal{A}_i(\theta,\gamma_i)\boldsymbol{\eta}(\theta)\\ &= -P_i^{-1}(\theta)P_i'(\theta)\boldsymbol{\chi}(\theta) + P_i^{-1}(\theta)\mathcal{A}_i(\theta,\gamma_i)P_i'(\theta)\boldsymbol{\chi}(\theta)\\ &= P_i^{-1}(\theta)[-P_i'(\theta) + \mathcal{A}_i(\theta,\gamma_i)P_i'(\theta)]\boldsymbol{\chi}(\theta)\\ &= P_i^{-1}(\theta)\{P_i(\theta)[A_{ci}(\theta) + \tfrac{1}{2}\gamma_i I_4]\}\boldsymbol{\chi}(\theta)\\ &= [A_{ci}(\theta) + \tfrac{1}{2}\gamma_i I_4]\boldsymbol{\chi}(\theta) \quad \text{(A17)}\end{aligned}$$

As a Lyapunov transformation does not change the characteristic multiplier set of a periodic system (see, for example, [34]), it follows from Eqs. (A13) and (A17) that

$$\mathcal{C}[A_{ci}(\theta) + \tfrac{1}{2}\gamma_i I_4] = \mathcal{C}[\mathcal{A}_i(\theta,\gamma_i)] = \{e^{-\pi\gamma_i}, e^{-\pi\gamma_i}, e^{-\pi\gamma_i}, e^{-\pi\gamma_i}\}$$

which indicates that the periodic system $\boldsymbol{\varphi}'(\theta) = [A_{ci}(\theta) + \frac{1}{2}\gamma_i I_4]\boldsymbol{\varphi}(\theta)$ is asymptotically stable. Hence, it follows from Lemma 6 that the periodic Lyapunov differential equation given by Eq. (A15) has a unique ω-periodic positive definite solution $[\partial P_i(\theta)]/\partial\gamma_i > 0$.

As $[\partial P_i(\theta)]/\partial\gamma_i > 0$, the limit of $P_i(\theta)$ as γ_i approaches zero exists. Denote

$$\lim_{\gamma_i\to 0^+} P_i(\theta) = P_{i0}(\theta) \geq 0$$

Taking the limit of both sides of Eq. (26) gives

$$\begin{aligned}-P_{i0}'(\theta) &= A_i^T(\theta)P_{i0}(\theta) + P_{i0}(\theta)A_i(\theta)\\ &- P_{i0}(\theta)B_i(\theta)R_i^{-1}(\theta)B_i^T(\theta)P_{i0}(\theta) \quad \text{(A18)}\end{aligned}$$

It is well known that (see, for example, [39]) if the periodic matrix pair $[A_i(t), B_i(t)]$ is controllable and all the characteristic multipliers of $A_i(\theta)$ are inside or on the unit circle, the periodic Riccati differential equation given in Eq. (A18) has the unique periodic nonnegative definite solution $P_{i0}(\theta) = 0$; namely,

$$\lim_{\gamma_i \to 0^+} P_i(\theta) = 0$$

The proof is completed.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under grant numbers 60904007 and 60928008, and 61074111, the China Postdoctoral Science Foundation under grant number 20100480059, the Heilongjiang Postdoctoral Foundation of China under grant number LRB10-194, the Foundation for Innovative Research Group of the National Natural Science Foundation of China under grant 601021002, the Development Program for Outstanding Young Teachers at the Harbin Institute of Technology under grant number HITQNJS.2009.054, and a Zhiyuan Chair Professorship at Shanghai Jiao Tong University, Shanghai, People's Republic of China.

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